MA2160
Test 3
Spring 2007

Name:________________________________

Instructions: You may NOT use your calculator on the entire test. You must show enough work to justify all answers.

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1. Find the sum of the geometric series $12 - 6 + 3 - \frac{3}{2} + \frac{3}{4} - \cdots$.

   Solution.

$$12 + 12\left(-\frac{1}{2}\right) + 12\left(-\frac{1}{2}\right)^2 + 12\left(-\frac{1}{2}\right)^3 + \cdots = \frac{12}{1-(-\frac{1}{2})} = 8$$

□

8 [10]

2. Find the third degree Taylor polynomial for $\sqrt{1+x}$ near $a = 3$.

   Solution.

$$\begin{array}{ll} f(x) = (1+x)^{1/2} & f(3) = 2 \\ f'(x) = \frac{1}{2}(1+x)^{-1/2} & f'(3) = \frac{1}{4} \\ f''(x) = \frac{1}{2}(-\frac{1}{2})(1+x)^{-3/2} & f''(3) = -\frac{1}{32} \\ f'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(1+x)^{-5/2} & f'''(3) = \frac{3}{256} \end{array}$$

$$2 + \frac{(x-3)}{4} - \frac{(x-3)^2}{32 \cdot 2!} + \frac{3(x-3)^3}{256 \cdot 3!}$$

□

$\sqrt{1+x} \approx 2 + \frac{(x-3)}{4} - \frac{(x-3)^2}{64} + \frac{(x-3)^3}{512}$ [10]

3. Find the Taylor series about $x = 0$ for $\frac{1}{(1-x)^2}$ from the series for $\frac{1}{1-x}$. You must write at least four terms explicitly.

Solution.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$\square$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ [15]

4. (a) Complete the following Theorem. [5]

Suppose f and all its derivatives are continuous. If $P_n(x)$ is the n^{th} Taylor polynomial for $f(x)$ about a, then

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $\max |f^{n+1}| \leq M$ on the interval between a and x.

(b) Estimate the magnitude of the error in approximating the following quantity using a third-degree Taylor polynomial about $x = 0$.

$$\sin 2$$

Solution.

$$|f^{(4)}| \leq 1$$

$$|E_3| \leq \frac{|2|^4}{4!} = \frac{2}{3}$$

□

$|E_3| < \frac{2}{3}$ [15]

5. Find the value(s) of ω for which $y = \cos(\omega t)$ satisfies

$$\frac{d^2y}{dt^2} + 9y = 0.$$

Solution.
$y' = -\omega \sin(\omega t)$
$y'' = -\omega^2 \cos(\omega t)$

$$\begin{aligned} -\omega^2 \cos(\omega t) + 9(\cos(\omega t)) &= 0 \\ (-\omega^2 + 9)(\cos(\omega t)) &= 0 \\ -\omega^2 + 9 &= 0 \\ \omega^2 &= 9 \\ \omega &= \pm 3 \end{aligned}$$

$\square$

$\omega = \pm 3$ [15]

6. Find the solution to the following differential equation with initial condition $z(1) = 5$.

$$\frac{1}{z}\frac{dz}{dt} = 5$$

Solution.

$$\begin{aligned} \frac{dz}{z} &= 5\ dt \\ \int \frac{dz}{z} &= \int 5\ dt \\ \ln|z| &= 5t + C \\ |z| &= Ae^{5t} \\ z &= Be^{5t} \end{aligned}$$

$$\begin{aligned} 5 &= Be^{5} \\ B &= 5e^{-5} \end{aligned}$$

$$\begin{aligned} z(t) &= 5e^{-5}e^{5t} \\ z(t) &= 5e^{5t-5} \end{aligned}$$

$\square$

$z(t) = 5e^{5t-5}$ [15]

7. Use Euler's method to approximate the value of y at $x = 1$ on the solution curve to the differential equation

$$\frac{dy}{dx} = y + 1$$

that passes through (0,0). Use $\Delta x = \frac{1}{3}$ (i.e., 3 steps).

Solution.

x	y	m	Δy
0	0	1	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{9}$
$\frac{2}{3}$	$\frac{7}{9}$	$\frac{16}{9}$	$\frac{16}{27}$
1	$\frac{37}{27}$	$\frac{64}{27}$	$\frac{64}{81}$

$\square$

$y \approx \frac{37}{27}$ [15]